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Solutions of Yang–Mills equations on generalized Hopf bundles ☆

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Abstract

Trautman has constructed natural self-dual connections on the Hopf bundles over complex and quaternionic projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$; the associated connections are $SU(n+1)$ and $Sp(n+1)$ invariant. Trautman wondered if these connections could be generalized to the case of the corresponding projective spaces defined by indefinite metrics. In this note, we extend the work of Trautman in two different directions. We first define self-dual connections on the Hopf bundles over the projective spaces $\mathbb{C}P^{(p,q)}$ and $\mathbb{H}P^{(p,q)}$ which are $U(p,q+1)$ and $Sp(p,q+1)$ invariant. We also define self-dual connections over the Hopf bundles associated with the para-complex and para-quarternionic projective spaces $\tilde{\mathbb{C}}P^{(p,q)}$ and $\tilde{\mathbb{H}}P^{(p,q)}$. Finally, the topology of these projective spaces is investigated. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let G be a compact simple Lie group. We use G as the gauge group in Yang–Mills theory. Let F be the field strength (curvature) of a connection A on a principle G bundle. The Yang–Mills field equation takes the form $\mathrm{D}^*F = 0$. If the underlying manifold of the theory is an oriented four-dimensional manifold, then we let $*$ be the associated Hodge operator. We say that the field strength F is self-dual or anti-self dual if ${}^*F = \pm F$. If

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F is self or anti-self dual, then the Bianchi identity $\mathrm{D}F = 0$ implies that the Yang–Mills equation is satisfied.

In this paper, we generalize solutions of Yang–Mills equations, which were given by Trautman [5], to the case of some interesting non-compact Lie groups. Here is a brief outline to this paper. In Section 2, we shall discuss the properties of the Clifford algebras $C(1, 0)$, $C(0, 1)$, $C(2, 0)$ and $C(0, 2)$ and the corresponding projective spaces which they define. In Section 3, we shall define the generalized Hopf fiberings. We shall compute the homotopy groups of the associated projective spaces and in some cases, we shall find the corresponding homotopy equivalence. In Section 4, we shall define self-dual connections on the generalized Hopf fiberings which are $U_{\mathbb{F}}(p, q + 1)$ or $Sp_{\mathbb{F}}(p, q + 1)$ invariant. We conclude in Section 5 by giving local representations of these connections.

2. The Clifford algebras and the projective spaces

We shall study the projective spaces which are defined not only by the complex numbers $\mathbb{C}$ and by the quaternions $\mathbb{H}$, but which, more generally, are defined by certain Clifford algebras. Let $\mathbb{R}^{(p,q)}$ be Euclidean space with an inner product η of signature (p, q), i.e., there exists a basis $v_1, \ldots, v_{p+q}$ of $\mathbb{R}^{(p,q)}$, such that $(v_\mathrm{i}, v_\mathrm{j}) = 0$ for $\mathrm{i} \neq \mathrm{j}$, $(v_\mathrm{i}, v_\mathrm{i}) = -1$ for $\mathrm{i} = 1, \ldots, p$, and $(v_\mathrm{i}, v_\mathrm{i}) = +1$ for $\mathrm{i} = p + 1, \ldots, p + q$. The Clifford algebra $C(p, q)$ is the universal unital real algebra which is generated by $\mathbb{R}^{(p,q)}$ subject to the relations

$$v * w + w * v = 2\eta(v, w), \quad v, w \in \mathbb{R}^{(p,q)}.$$

We note that

$$C(1, 0) = \mathbb{C} \text{ and } C(2, 0) = \mathbb{H}$$

are the algebras of the complex numbers and of the quaternions, respectively. The algebras

$$C(0, 1) = \tilde{\mathbb{C}} \text{ and } C(1, 1) = C(1, 2) = \tilde{\mathbb{H}}$$

are the algebras of the para-complex and of the para-quaternionic numbers. We can use the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\tilde{\mathbb{C}}$, and $\tilde{\mathbb{H}}$ to define projective spaces $\mathbb{R}P^{(p,q)}$, $\mathbb{C}P^{(p,q)}$ and $\mathbb{H}P^{(p,q)}$ (see [3,6]) and $\tilde{\mathbb{C}}P^{(p,q)}$ and $\tilde{\mathbb{H}}P^{(p,q)}$, respectively (see [1,2] for $\tilde{\mathbb{C}}P^{(0,q)}$ and $\tilde{\mathbb{H}}P^{(0,q)}$).

We shall now review the important properties of the algebras $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{H}}$ and define the associated projective spaces $\tilde{\mathbb{C}}P^{(p,q)}$ and $\tilde{\mathbb{H}}P^{(p,q)}$. The real Clifford algebra $\tilde{\mathbb{C}} := C(0, 1)$ of para-complex numbers is generated by 1 and by a generator i which satisfies the relation $\mathrm{i}^2 = 1$. Thus

$$(a + \mathrm{i}b)(c + \mathrm{i}d) = (ac + bd) + \mathrm{i}(ad + bc).$$

Let $f = x + \mathrm{i}y \in \tilde{\mathbb{C}}$. As in the case of complex numbers, we define

$$\bar{f} = x - \mathrm{i}y, \quad \Re f = x, \quad \Im f = \mathrm{i}y \quad \text{and } |f|^2 = f\bar{f} = x^2 - y^2.$$

Note that $|f|^2$ can be non-negative or zero — there are isotropic elements $f \neq 0 \in \tilde{\mathbb{C}}$, such that $|f|^2 = 0$. Similarly, let $\tilde{\mathbb{H}} := C(1,1) = C(0,2)$ be the para-quaternions. The algebra $\mathbb{H}$ is generated by 1, i, j, k subject to the relations

$$-\mathrm{i}^2 = 1 = \mathrm{j}^2 = \mathrm{k}^2 \quad \text{and} \quad \mathrm{ij} = \mathrm{k} = -\mathrm{ji}.$$

If $f = x + \mathrm{i}y + \mathrm{j}z + \mathrm{k}w$, then we define

$$\bar{f} := x - \mathrm{i}y - \mathrm{j}z - \mathrm{k}w, \qquad \mathfrak{R}f := x, \qquad \mathfrak{J}f := \mathrm{i}y + \mathrm{j}z + \mathrm{k}w, \qquad |f|^2 := f\bar{f}.$$

Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \tilde{\mathbb{C}}, \tilde{\mathbb{H}}\}$ be one of these four algebras. To have a common notation, we set

$$\mathrm{i}^2 = -\epsilon_1, \qquad \mathrm{j}^2 = -\epsilon_2, \qquad \mathrm{k}^2 = -\epsilon_3,$$

where

$$\epsilon_1 = 1 \text{ if } \mathbb{F} = \mathbb{C}, \qquad \epsilon_1 = -1 \text{ if } \mathbb{F} = \tilde{\mathbb{C}},$$
$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 1 \text{ if } \mathbb{F} = \mathbb{H}, \qquad \epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1 \text{ if } \mathbb{F} = \tilde{\mathbb{H}}.$$

We say that a vector is space-like if $|v|^2 > 0$. Since $|fh|^2 = |f|^2|h|^2$ for all of these algebras, the set of unit spacelike vectors in $\mathbb{F}$ forms a group under multiplication; let $\mathbb{F}_1$ be the connected component of the identity in this group. Let $S^{(p,q)}$ be the pseudosphere of unit spacelike vectors in $R^{(p,q)}$. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. We then have

$$\mathbb{C}_1 = U(1) = \mathrm{Spin}(2) = S^1, \qquad \mathbb{H}_1 = \mathrm{Spin}(3) = SU(2) = S^3,$$
$$\tilde{\mathbb{C}}_1 = \{x + \mathrm{i}y \in \bar{\mathbb{C}} : x^2 - y^2 = 1, +y0\} = \mathrm{Spin}(0,1)_0,$$
$$\tilde{\mathbb{H}}_1 = \{x + \mathrm{i}y + \mathrm{j}z + \mathrm{k}w \in \bar{\mathbb{H}} : x^2 + y^2 - z^2 - w^2 = 1\} = S^{(2,1)} = SU(1,1)$$
$$= \mathrm{Spin}(1,2).$$

Let $n = p + q$. Let $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$ be elements of $\mathbb{F}^{n+1}$. Let $d := \dim_{\mathbb{R}}\mathbb{F}$. We define a pseudo-Riemannian metric $g = g_{(p,q+1)}$ on $\mathbb{F}^{n+1}$ by

$$g_{(p,q+1)}(x,y) := \mathfrak{R}(-x_1\bar{y}_1 - \cdots - x_p\bar{y}_p + \cdots + x_{n+1}\bar{y}_{n+1}),$$
$$\mathrm{signature}(g_{(p,q+1)}) = (dp, d(q+1)) \text{ if } \mathbb{F} \in \{\mathbb{C}, \mathbb{H}\},$$
$$\mathrm{signature}(g_{(p,q+1)}) = (\tfrac{1}{2}d(n+1), \tfrac{1}{2}d(n+1)) \text{ if } \mathbb{F} \in \{\tilde{\mathbb{C}}, \tilde{\mathbb{H}}\}.$$

Let $\mathbb{F}^{(p,q+1)} := (\mathbb{F}^{n+1}, g_{p,q+1})$. If $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, then we let $\mathbb{F}P^{(p,q)}$ be the set of spacelike lines in $\mathbb{F}^{(p,q+1)}$, where $\mathbb{F}$ acts by scalar multiplication from the right. However, it is necessary to be a bit more careful if $\mathbb{F} \in \{\tilde{\mathbb{C}}, \tilde{\mathbb{H}}\}$. Let $\mathbb{F}^+ := \{f \in \mathbb{F} : |f|^2 > 0\}$ be the group of spacelike elements of $\mathbb{F}$ and let $\mathbb{F}_0^+$ be the connected component of the identity; $\mathbb{F}_1 = \{f \in \mathbb{F}_0^+ : |f|^2 = 1\}$. We say $x \in \mathbb{F}^{(p,q+1)}$ is spacelike if $g(x,x) > 0$. We say that two spacelike vectors x and y are equivalent if there exists $f \in \mathbb{F}_0^+$, so that $xf = y$. Let $[x]$ denotes the equivalence class defined by x and let $\mathbb{F}P^{(p,q)}$ be the set of these equivalence classes. The projective spaces

$$\{\mathbb{C}P^{(p,q)}, \ \mathbb{H}P^{(p,q)}, \ \tilde{\mathbb{C}}P^{(p,q)}, \ \tilde{\mathbb{H}}P^{(p,q)}\}$$

inherit natural differentiable structures and pseudo-Riemannian metrics of signatures $(2p, 2q)$, $(4p, 4q)$, (n, n), and $(2n, 2n)$, respectively. Notice that $\mathbb{C}P^{(0,q)} = \mathbb{C}P^q$ and $\mathbb{H}P^{(0,q)} = \mathbb{H}P^q$ are the standard complex and quaternionic projective spaces, respectively. The spaces $\mathbb{C}P^{(2p,2q)}$ and $\mathbb{H}P^{(4p,4q)}$ were studied by Wolf [6]. In the positive definite setting, the projective spaces defined by Clifford algebras $C(0, 1) = \tilde{\mathbb{C}}$ and $C(0, 2) = \tilde{\mathbb{H}}$ were considered in [1] and their sectional curvatures were determined.

3. The generalized Hopf fiberings

Let $S = S(\mathbb{F}^{(p,q+1)})$ be the pseudosphere of unit space-like vectors

$$S(\mathbb{F}^{(p,q+1)}) := \{v \in \mathbb{F}^{(p,q+1)} : \langle v, v\rangle = 1\}.$$

The map

$$\pi : S(\mathbb{F}^{(p,q+1)}) \to \mathbb{F}P^{(p,q)}, \quad \pi(x) = [x]$$

defines a principal bundle with structure group $\mathbb{F}_1$ over the projective space $\mathbb{F}P^{(p,q)}$

Total space S is the unit pseudosphere $S^{(dp,d(q+1)-1)}$ of the signature $(dp, d(q + 1) - 1)$ which is diffeomorphic to

$$R^{dp} \times S^{d(q+1)-1} \approx \mathbb{F}^p \times S^{d(q+1)-1}.$$

Let us study the homotopy groups of the projective spaces $\mathbb{F}P^{(p,q)}$. Because of the generalized Hopf fibration,

$$\mathbb{F}_1 \to S \to \mathbb{F}P^{(p,q)},$$

we have followed the long exact sequence of the corresponding homotopy groups

$$\cdots \to \pi_m(\mathbb{F}_1) \to \pi_m(S) \to \pi_m(\mathbb{F}P^{(p,q)}) \to \pi_{m-1}(\mathbb{F}_1) \to \cdots .$$

Hence, the homotopy groups of $\mathbb{F}P^{(p,q)}$ are

$$\pi_m(\mathbb{C}P^{(p,q)}) = \pi_m(\mathbb{C}P^q), \qquad \pi_m(\mathbb{H}P^{(p,q)}) = \pi_m(\mathbb{H}P^q),$$
$$\pi_m(\tilde{\mathbb{C}}P^{(p,q)}) = \pi_m(S^{2q+1}), \qquad \pi_m(\tilde{\mathbb{H}}P^{(p,q)}) = \pi_m(\mathbb{C}P^{2q+1})$$

for all $m \in \mathbb{N}$. We are going to show that in the first two cases, there exist homotopy equivalence $\mathbb{C}P^{(p,q)} \simeq \mathbb{C}P^q$ and $\mathbb{H}P^{(p,q)} \simeq \mathbb{H}P^q$. Denote by $\mathbb{F}$ complex numbers or quaternions. The mapping

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_p, x_{p+1}\lambda, \ldots, x_{p+q+1}\lambda),$$
$$\lambda = (1 + |x_1|^2 + \cdots + |x_p|^2)^{1/2}$$

is a diffeomorphism of pseudosphere S and $\mathbb{F}^p \times S^{d(q+1)-1}$. The mappings

$$f : \mathbb{F}P^{(p,q)} \to \mathbb{F}P^q, \qquad g : \mathbb{F}P^q \to \mathbb{F}P^{(p,q)}$$

defined by

$$f([(x_1, \dots, x_{n+1})]) := [(x_{p+1}\lambda, \dots, x_{p+q+1}\lambda)]$$

and

$$g([(x_1, \dots, x_{q+1})]) := [(0, \dots, 0, x_1, \dots, x_{q+1})]$$

are well defined and continuous. Moreover, there are homotopies

$$f \circ g \simeq Id_{\mathbb{F}P^{(p,q)}}, \qquad g \circ f \simeq Id_{\mathbb{F}P^q}$$

and hence spaces $\mathbb{F}P^{(p,q)}$ and $\mathbb{F}P^q$ are homotopically equivalent in the case of the complex numbers and the quaternions. At the moment, we do not know if there is a similar homotopy equivalence in the case of the Clifford algebras $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{H}}$.

4. Self-dual connections

In this section, we shall construct a natural connection on the generalized Hopf fiberings and prove that this connection is self-dual. The Lie algebra $\mathcal{F}$ of the group $\mathbb{F}_1$ is algebra of imaginary numbers of $\mathbb{F}$. Let σ be the map from $\mathcal{F}$ onto corresponding fundamental field tangent to S; the value at $x = (x_1, \dots, x_{n+1}) \in S$ is given by

$$\sigma(X)(x) = (x_1 X, \dots, x_{n+1} X), \quad X \in \mathcal{F}.$$

We follow the discussion in [4]. In order to define the connection on generalized Hopf bundle, we must find an $\mathcal{F}$-valued one form ω on the total space S which satisfies the following conditions:

1. $\omega(\sigma(X)) = X, \quad x \in \mathcal{F}.$
2. $\omega(R_{f*}(Y)) = Ad(f^{-1})\omega(Y), \qquad f \in \mathbb{F}_1, \qquad Y \in TS.$

We consider the following differential form given at a point $x = (x_1, \dots, x_{n+1})$ by

$$\omega(x_1, \dots, x_{n+1}) := -\bar{x}_1\, dx_1 - \dots - \bar{x}_p\, dx_p + \dots + \bar{x}_{n+1}\, dx_{n+1} = \bar{x}\, dx.$$

This differential form defines a connection on the total space S which is $U_{\mathbb{F}}(p, q+1)$ or $Sp_{\mathbb{F}}(p, q+1)$ invariant. By $U_{\mathbb{F}}(p, q+1)$ (respectively $Sp_{\mathbb{F}}(p, q+1)$), we have denoted the subgroups of $Gl_{d(n+1)}(\mathbb{R})$ of the endomorphisms which commute with the right action of $\mathbb{F} \in \{\mathbb{C}, \tilde{\mathbb{C}}\}$ (respectively $\mathbb{F} \in \{\mathbb{H}, \tilde{\mathbb{H}}\}$) on $\mathbb{R}^{d(n+1)} = \mathbb{F}^{n+1}$, and which preserve the scalar product $g_{(p,q+1)}$.

Let D be the covariant derivative defined by the connection ω and let φ be a horizontal form. We then have

$$\mathrm{D}\varphi = \mathrm{d}\varphi + \omega \wedge \varphi.$$

Consequently, the curvature two form Ω of this connection is given by

$$\Omega = \mathrm{D}\omega = \mathrm{d}\omega + \omega \wedge \omega. \tag{1}$$

Furthermore, we have the Bianchi identity $\mathrm{D}\Omega = 0$. Let $s : \mathbb{F}P^{(p,q)} \to S$ be a section to the generalized Hopf bundle. Let $\omega_s = s^*\omega$, $\Omega_s := s^*\Omega$, and $\mathrm{D}_s\varphi = \omega_s \wedge \varphi$ be the pull-back of the connection 1-form, the curvature 2-form, and the covariant derivative, respectively.

The main result of the paper is the following.

Theorem 4.1. *The curvature form Ω_s is source-free, i.e., satisfies $\mathrm{D}_s^*\Omega_s = 0$ for every section s. The connection ω is $U_{\mathbb{F}}(p, q+1)$ or $Sp_{\mathbb{F}}(p, q+1)$ invariant.*

Proof. The vertical subspaces of the tangent space T_xS for $x \in S$ are generated by the vectors $x\mathrm{i}, x\mathrm{j}, x\mathrm{k}$; these spaces do not depend on the particular connection which is chosen. We use the connection ω to induce a splitting $TS = TS_\mathrm{h} \oplus TS_\mathrm{v}$ of tangent bundle of total space S into horizontal and vertical part. We now show that the horizontal part X_h of any $X \in T_xS$ is given by

$$X_\mathrm{h} := X + x(\bar{X}x).$$

In fact, we have

$$\omega(x)(X) = \bar{x}(X + x(\bar{X}x)) = \bar{x}X + \bar{X}x = 2g(x, X) = 0,$$

since $X \in T_xS$. Every $X_\mathrm{h} \in T_xS$ is the horizontal lift of some vector $u \in T_{[x]}\mathbb{F}P^{(p,q)}$, which is given by $u = \pi_*(X_\mathrm{h}) = \pi_*(X)$.

Notice that for any $X, Y \in T_xS$, we have

$$\bar{X}_\mathrm{h}Y_\mathrm{h} = \bar{X}Y + \omega(x)(X)\omega(x)(Y).$$

Now, using definition given above, we obtain the following expression for the curvature form Ω

$$\Omega(X, Y) = \bar{X}_\mathrm{h}Y_\mathrm{h} - \bar{Y}_\mathrm{h}X_\mathrm{h} = 2\Im(\bar{X}_\mathrm{h}Y_\mathrm{h}), \quad X, Y \in T_xS.$$

Let $X_\mathrm{h}, Y_\mathrm{h} \in T_xS$ be any horizontal lifts of u and v. Let $\tilde{g}$ and F be the induced metric and the curvature form on projective space

$$\tilde{g}(u, v) := g(X_\mathrm{h}, Y_\mathrm{h}) = \Re(\bar{X}_\mathrm{h}Y_\mathrm{h}), \quad u, v \in T_{[x]}\mathbb{F}P^n,$$
$$F(u, v) := \Omega_s(X_\mathrm{h}, Y_\mathrm{h}) = \Omega_s(X, Y), \quad u, v \in T_{[x]}\mathbb{F}P^n.$$

These definitions depend on the section s chosen.

We define fundamental forms on $\mathbb{F}P^n$ by

$$h_1(u, v) := \tilde{g}(u, v\mathrm{i}), \qquad h_2(u, v) := \tilde{g}(u, v\mathrm{j}),$$
$$h_3(u, v) := \tilde{g}(u, v\mathrm{k}), \quad u, v \in T\mathbb{F}P^n.$$

The tangent vectors $v\mathrm{i}, v\mathrm{j}, v\mathrm{k}$ on $\mathbb{F}P^n$ are locally well defined by

$$v\mathrm{i} := \pi_*(Y\mathrm{i}), \qquad v\mathrm{j} := \pi_*(Y\mathrm{j}), \qquad v\mathrm{k} := \pi_*(Y\mathrm{k}),$$

where tangent vectors v and Y are related as above. The curvature form Ω can be written in the form

$$F = 2\epsilon_1 h_1 \mathrm{i}$$

in the case of the complex and para-complex numbers and in the form

$$F = 2(\epsilon_1 h_1 \mathrm{i} + \epsilon_2 h_2 \mathrm{j} + \epsilon_3 h_3 \mathrm{k})$$

in the case of the quaternions and para-quaternions. Notice that in the latter case, we have the real form

$$F \wedge F = 4(\epsilon_1 h_1 \wedge h_1 + \epsilon_2 h_2 \wedge h_2 + \epsilon_3 h_3 \wedge h_3),$$

which does not depend on the section s. In the case of the complex and para-complex numbers, the $2n$-form $F \wedge \cdots \wedge F$ (n factors) and similarly, in the quaternionic and para-quaternionic case, the $4n$-form $F \wedge \cdots \wedge F$ ($2n$ factors) is up to a constant scalar volume form on $\mathbb{F}P^{(p,q)}$.

Hence, for any fixed section s, there is constant λ, such that

$$^*F([x]) = \lambda \underbrace{F \wedge \cdots \wedge F}_{(nd/2)-1}.$$

We use the Bianchi identity to see that $\mathrm{D}_s^* \Omega_s = 0$. □

We remark that $U_{\tilde{\mathbb{C}}}(0, n+1)$ is isomorphic to $GL(n+1, \mathbb{R})$ (see [2]). Moreover, $Sp_{\tilde{\mathbb{H}}}(0, n) = Sp(n, \mathbb{R})$ and $Gl_n(\tilde{\mathbb{H}}) = Gl_{2n}(\mathbb{R})$.

5. Local representation of connection

We now construct local representatives of the connection ω, and show that in some low-dimensional cases, these correspond to the solutions of Maxwell and Yang–Mills equations. Let

$$U_k := \{(x_1 : \cdots : x_{n+1}) \in \mathbb{F}P^{(p,q)} | x_k \neq 0\}, \quad k = 1, \ldots, n+1$$

be the standard chart of $\mathbb{F}P^{(p,q)}$. On

$$\pi^{-1}(U_k) = \{(x_1, \ldots, x_{n+1}) \in S \subset \mathbb{F}^{p,q+1} | x_k \neq 0\},$$

we introduce local trivializations t_k of our generalized Hopf bundle, given by

$$t_k := (\pi, \varphi_k) : \pi^{-1}(U_k) \to U_k \times \mathbb{F}_1,$$

where

$$\varphi_k(x_1, \ldots, x_{n+1}) := \frac{x_k}{|x_k|} \in \mathbb{F}_1.$$

We may represent a point $[x] = (x_1 : \cdots : x_{n+1}) \in \mathbb{F}P^{(p,q)}$ in a chart U_{n+1} in the form

$$\xi = (\xi_1, \ldots, \xi_n) = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \in \mathbb{F}^{p,q},$$

so

$$t_k^{-1}(\xi, m) = \left(\frac{\xi \xi_k^{-1} |\xi_k| m}{\sqrt{1 + |\xi|^2}}, \frac{\xi_k^{-1} |\xi_k| m}{\sqrt{1 + |\xi|^2}}\right).$$

We define local sections $s_k : U_{n+1} \cap U_k \to \pi^{-1}(U_k)$ by setting

$$s_k(\xi) := t_k^{-1}(\xi, 1).$$

By direct calculation, we find a local representative of the connection ω

$$\omega_{n+1}(\xi)Y := s_{n+1}^{*}\omega = \frac{\bar{\xi}Y - \bar{Y}\xi}{2(1+|\xi|^2)},$$

where $Y \in T_\xi \mathbb{F}^{p,q}$ is representative of a vector tangent to $\mathbb{F}P^{(p,q)}$ and the following notation is used

$$\bar{\xi}Y = \bar{\xi}_1 Y_1 \cdots \bar{\xi}_n Y_n.$$

Noticing that $\gamma = \gamma_{k(n+1)} = \xi_k/|\xi_k|$ is the transition function, from section s_k, to section s_{n+1}, we have

$$\omega_k(\xi)(Y) := s_k^{*}\omega = \gamma^{-1}\,\mathrm{d}\gamma + \gamma^{-1}\omega_{n+1}\gamma = \frac{\xi\bar{Y} - Y\bar{\xi}}{2|\xi|^2(1+|\xi|^2)}.$$

In case of positive signature $\mathbb{F} = \mathbb{C}$, $p = 0, q = 1$, these are Maxwell solutions, while in case of $\mathbb{F} = \mathbb{H}$, $p = 0, q = 1$, these are Yang–Mills solutions which are usually written using Pauli matrices.

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